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On the dynamical and geometrical symmetries of Keplerian motion

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Abstract

The dynamical symmetries of classical, relativistic and quantum-mechanical Kepler systems are considered to arise from geometric symmetries in PQET phase space. To establish their interconnection, the symmetries are related with the aid of a Lie-algebraic extension of Dirac's correspondence principle, a canonical transformation containing a Cunningham–Bateman inversion, and a classical limit involving a preliminary canonical transformation in ET space. The Lie-algebraic extension establishes the conditions under which the uncertainty principle allows the local dynamical symmetry of a quantum-mechanical system to be the same as the geometrical phase-space symmetry of its classical counterpart. The canonical transformation converts Poincaré-invariant free-particle systems into ISO(3,1) invariant relativistic systems whose classical limit produces Keplerian systems. Locally Cartesian relativistic PQET coordinates are converted into a set of eight conjugate position and momentum coordinates whose classical limit contains Fock projective momentum coordinates and the components of Runge–Lenz vectors. The coordinate systems developed via the transformations are those in which the evolution and degeneracy groups of the classical system are generated by Poisson-bracket operators that produce ordinary rotation, translation and hyperbolic motions in phase space. The way in which these define classical Keplerian symmetries and symmetry coordinates is detailed. It is shown that for each value of the energy of a Keplerian system, the Poisson-bracket operators determine two invariant functions of positions and momenta, which together with its regularized Hamiltonian, define the manifold in six-dimensional phase space upon which motions evolve.

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1. Introduction

Current understandings of dynamical symmetries of Kepler systems might be said to begin with Fock's 1935 discovery of the hyperspherical symmetry of hydrogen-like atoms [1]. It was immediately followed by Bargmann's production of the first- and second-order differential operators whose action on energy-degenerate Schrödinger wavefunctions is isomorphic to that of generators of the Lie groups SO(4), SO(3,1) and E(3) [2]. The introduction of SU(3) into elementary particle physics in the early 1960s catalyzed a vast resurgence of interest in these Lie symmetries. This led to the discovery of the SO(4,1) and SO(4,2) 'hidden' symmetries of Kepler systems, the Lie symmetries of a host of other systems, the development of the concept of 'dynamical symmetry', and the development of the theory of the generalized Lie (Lie–Backlund) transformations required to properly formulate many of the invariance transformations of Schrödinger's (and other) partial differential equations. Review articles by Mclntosh [3], and by Bander and Itzykson [4] describe the many investigations of dynamical symmetries published during this period. Further references will be found in Barut's 1971 monograph [5], and in a paper by Evans published in 1990 [6]. The SO(4,1) dynamical symmetry of hydrogen-like atoms was discovered in 1966 [7], as was their SO(4,2) symmetry [8]. The SO(4,2) dynamical symmetry of classical Kepler systems was established by Gyorgyi in 1967 [9].

Many textbooks and monographs now deal with the dynamical symmetries of hydrogen-like atoms [10]. Bluman and Kumei review and extend work on Lie–Backlund symmetries in their 1989 textbook, *Symmetries and Differential Equations* [11]. Further developments are contained in Cantwell's *Introduction to Symmetry Analysis* [12], published in 2002, and in a series of monographs by Ibragimov [13].

In the 1970s, interest in relativistic quantum field theories led to a resurgence of interest in the invariance of Maxwell's equations under the inversion that had been discovered by Bateman and Cunningham in 1909 [14]. Thousands of papers exploring the physical significance and the mathematics of the resulting projective and conformal groups have been published, many of them dealing with the role of the group in other fields (e.g., statistical mechanics). Kastrup has written two helpful overviews of these developments [15]. In their monograph, *Theory of Group Representations and Applications* [16]. Barut and Raczka have given a straightforward discussion of relativistic wave equations invariant under the Bateman–Cunningham conformal group.

A 1990 article by Shlomo Sternberg and Victor Guillemin, 'Variations on a Theme by Kepler' provides an overview of many mathematical connections relevant to studies of Keplerian systems, not a few of which have arisen from such studies [17].

In the current contribution, the dynamical symmetries of a relativistic, and a quantum-mechanical, Kepler system are related to the dynamical symmetries of classical Kepler systems, which are considered to be geometrical symmetries in PQET phase space. The Lie algebraic extension of Dirac's correspondent principle that is mentioned in the abstract, and established in section 7, refines and extends an earlier version due to the present author [18].

To establish a clear relation between the dynamical symmetry of classical Kepler systems and the dynamical symmetry of the relativistic Kepler system, two canonical transformations are introduced. Toward the end of section 2, to properly deal with the fact that classical Kepler trajectories of different energies may have different topologies, the first of these, a canonical diffeomorphism in ET space is described. It is utilized in section 4, in which both closed and open classical trajectories become related to relativistic trajectories that are open curves. The transformation is based on the work of Poincaré [19], and Wulfman and Wybourne [20]. In section 3, a further, previously unexploited, canonical transformation is introduced. It is a Levi Civita transformation [21] in PQET space that contains a Bateman–Cunningham inversion. It interconverts two canonically conjugate sets of locally Cartesian position and momentum coordinates, (q^4, p^4) and (Q^4, P^4), and is a diffeomorphism in every region of PQET space that does not contain a light cone or its canonical conjugate. The transformation converts a Poincaré-invariant free-particle Hamiltonian into a Lorentz invariant version of a Kepler Hamiltonian, a Hamiltonian that has ISO(3,1) dynamical symmetry.

In section 4 it is shown that the classical limit, Q^c, P^c, of the Q^4 and P^4 contains the components of the Runge–Lenz vectors, $\mathbf{a}(E)$, and corresponding Fock projective momenta. In section 5, the eight Q^c, P^c, are shown to define a set of eight 'symmetry adapted' coordinates, four $\rho(E)_k$ and four $\pi(E)_k$. The four $\pi(E)_k$ are contained in an invariant PE subspace of PQET phase space. The set of variables $\rho(E)_k$ and $\pi(E)_k$ differs from similar variables introduced by Gyorgi [9] only by energy-dependent proportionality factors. Neither set of variables are canonically conjugate sets in either PQ, or PQET, space.

The relation of classical Kepler symmetries to symmetries and topologies in phase space has been investigated by a number of authors. In 1970, Moser used the classical analog of Fock's stereographic projection to establish the S(3) topology of negative energy Kepler manifolds in phase space [22]. In 1974, Souriau established that when the energy is negative, regularized Kepler manifolds in PQ phase space have the SO(4) symmetry, as well as the S(3) topology, of the surface of a hypersphere in four-dimensional Euclidean space [23]. In a paper published in 1977, Belbruno investigated unbounded Kepler motions in the plane, and proved that symplectic diffeomorphisms map the corresponding phase space manifolds for $E > 0$ onto Lobachevsky manifolds with curvature -1, and map the manifold for $E = 0$ onto a Euclidean plane punctured at the origin [24]. In the same year Osipov established that all Keplerian motions are equivalent to geodesic flows in spaces of constant curvature [25]. In the process he used a series of transformations, some of which when combined become nonrelativistic versions of the transformation which we introduce in section 3.

The transformations introduced in sections 2–5 connect, and produce, coordinate systems in which Keplerian motions of a representing point in phase space are produced by ordinary translations, rotations and hyperbolic rotations. They describe motions on manifolds in phase space that have self-evident symmetries determined by their invariance under such motions. The general results are not new; their representation in this simple manner is. In establishing it, a previously unknown result is also established: classical Kepler phase-space manifolds are, for every value of E, determined by motions restricted by three invariant functions, one of which is the Hamiltonian function. These functions are given explicitly. They determine the geometric expression of the dynamical symmetries of the well-known SO(4), SO(3,1) and $E(3)$ degeneracy groups, and the geometric expression of the SO(3,1) group that contains them. The connections of these symmetries to those of SO(4,2) are briefly described in section 6.

Before concluding the discussion of historical antecedents of the present work, its conceptual dependence on some much earlier work should be noted. The constants of motion of Kepler systems were known to Laplace [26], and to Hamilton [26]. During the second half of the nineteenth century, the development of new conceptual insights prepared paths to a deeper understanding of their implications. Sophus Lie's geometric intuitions guided the development of his theory of differential equations into a theory of the continuous groups that they define, and depend upon for their integrability [27]. During this period, several compound sentences were required to state the conditions now summarized in the term 'diffeomorphism'. In the same period, Gibb's study of Maxwell's equations led him to develop the geometric concept 'vector' [28]. Shortly thereafter, the aforementioned constants of Laplace and Hamilton were assembled into the vector Pauli named after Runge and Lenz [29], in his, the first published investigation of quantum-mechanical Kepler systems [30]. Milne's *Vectorial Mechanics*

[31] presents a treatment of Keplerian motion which was developed in this period. In it, orbits and hodo-graphs are obtained as scalar and vector products of the Runge–Lenz vector with, respectively, the position and momentum vectors of the moving body. Much later, in 1960, Synge pointed out that all of Hamiltonian mechanics admits a complete geometrization in PQET phase space [32]. From the latter standpoint, Hamilton's equations of motion determine the motion of a vector in phase space, and determine the manifolds that its terminus describes. From a Lie standpoint, Hamilton's equations define continuous evolution groups, and the invariance transformations of the equations define their invariance groups. The Lie generators of these evolution and invariance transformations in phase space are Poisson-bracket ('PB') operators. The transformations themselves are diffeomorphisms that leave invariant a symplectic metric and define local Lie transformation groups in symplectic space [33]. Two objects that can be interconverted by symplectic diffeomorphisms have the same Lie symmetry in symplectic space. The phase space of physics should be distinguished from symplectic space because general scale changes, which may be composed of uniform dilatations as well as symplectic dilatations $(q, p) \to (a\, q, p/a)$, are often used in establishing relations between observations and position and momentum variables. In this paper we suppose that this has already been done, and the symmetry labels are those of Lie groups of symplectic diffeomorphisms.

As the previous remarks suggest, geometrization of dynamical symmetries leads to questions of their physical interpretation. The following section sets forth some of the mathematical and logical relationships that will subsequently be used to address questions involved in relating geometrical symmetries of Kepler systems in phase space to the physical relationships codified in their dynamical groups.

2. Description of classical Kepler motions

We begin our treatment of Keplerian symmetries by further developing the nineteenth century idea of referring the motion to a coordinate system defined by its vector constants of motion. In terms of dimensionless variables in which the body's reduced rest mass, m_o, has unit value and the potential energy is $-K/r$, the seven well-known scalar constants of Kepler motion are the energy, E, the values of the components of the angular momentum vector, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, and the values of the components of the Runge–Lenz vector which we shall express as

$$\mathbf{a}(E) = (p^2 + 2E)\mathbf{r}/2 - (\mathbf{r} \cdot \mathbf{p})\mathbf{p}. \tag{1a}$$

The unit vectors

$$\mathbf{a}(E)/|\mathbf{a}(E)|, \qquad \mathbf{L}/|\mathbf{L}|, \qquad (\mathbf{a}(E)/|\mathbf{a}(E)|) \times (\mathbf{L}/|\mathbf{L}|), \tag{1b}$$

provide a convenient right-handed orthogonal unit triad and frame of reference to which one may refer the motion and its perturbations. Because

$$\mathbf{r} = (\mathbf{a}(E) - \mathbf{a}(-E))/|2E|, \tag{2}$$

for nonzero E, the motion of $\mathbf{a}(-E)$ with respect to $\mathbf{a}(E)$ describes the motion of $\mathbf{r}$. For $E = 0$, the motion of $\mathbf{p}$ with respect to $\mathbf{a}(0)$ determines the evolution of $\mathbf{r}$. It is convenient to set time, t, equal to zero at perhelion, at which $\mathbf{r} \cdot \mathbf{p} = 0$.

We will replace the equation $\mathrm{H}(\mathbf{q}, \mathbf{p}) - E = 0$, by its regularization,

$$r(\mathrm{H} - E) = H - K = 0, \tag{3a}$$

$$H = H(E) = r(p^2 - 2E)/2. \tag{3b}$$

Keplerian motion in the position space of $\mathbf{q} \equiv \mathbf{r}$ will be associated with the motion of points with coordinates ($\mathbf{q}$, t, $\mathbf{p}$, $-E$), ($\mathbf{q}$, $\mathbf{p}$, $-E$), ($\mathbf{q}$, $\mathbf{p}$) in the phase spaces PQET, PQE, PQ. The Poisson bracket ('PB') of two functions F, G in classical PQET space will be denoted by

$$\{F, G\} = \sum_1^4 (\partial F/\partial q_j \; \partial G/\partial p_j - \partial F/\partial p_j \; \partial G/\partial q_j), \tag{4a}$$

with $q_4 = t$, $p_4 = -E$. For reasons noted in the following paragraph, though dt represents a physical time interval, even in classical physics t may not necessarily represent Newtonian physical time. The Poisson-bracket operator of a differentiable function F will be denoted by

$$\{F\cdot\} = \sum_1^4 (\partial F/\partial q_j \; \partial/\partial p_j - \partial F/\partial p_j \; \partial/\partial q_j). \tag{4b}$$

It generates a one-parameter group of symplectic transformations, carried out by the operator

$$\exp(\alpha\{F\cdot\}), \tag{4c}$$

in which α is the group parameter. We will make considerable use of the operators in which F is $L_{ij} = (q_i p_j - q_j p_i)$, or $K_{ij} = (q_i p_j + q_j p_i)$. The PB operators of these functions are

$$\{L_{ij}\cdot\} = -((q_i\partial/\partial q_j - q_j\partial/\partial q_i) + (p_i\partial/\partial p_j - p_j\partial/\partial p_i)), \tag{4d}$$

$$\{K_{ij}\cdot\} = -((q_i\partial/\partial q_j + q_j\partial/\partial q_i) - (p_i\partial/\partial p_j + p_j\partial/\partial p_i)). \tag{4e}$$

If F is any C^2 function of p's and q's, then the first-order differential operator $\{F\cdot\}$ is capable of becoming the generator of a many-parameter Lie transformation group.

Let τ be an evolution parameter which ranges through all the reals [34]. Then Hamilton's equations of motion in the PQET space containing H are special cases of the equation

$$\mathrm{d}F(p, q)/\mathrm{d}\tau = \{F\cdot\}H = \{F, H\}. \tag{5}$$

Equations (3) define a six-dimensional 'Hamiltonian manifold' in PQE and PQET space, and a family of five-dimensional Hamiltonian manifolds in PQ phase space.

For the H of (3), equation (5) implies that d$t = r\,\mathrm{d}\tau$. For states with $E \geqslant 0$, both r and t range over an open interval, but for bound states, r is cyclic, so mathematically, t is cyclic, and confined within a compact interval. Both Hamiltonian and Lagrangian mechanics have this property of forcing t to repeat itself when $\mathbf{r}$ and $\mathbf{p}$ do so. They thereby force $\partial/\partial t$ to become the generator of a compact group [20]. For this reason we make use of a canonical transformation ('CT') which, for different values of E, creates topologically correct coordinates in the ET subspace of PQET space. For $E < 0$, $E > 0$, and, $0_- \leqslant E \leqslant 0_+$, define coordinates $p'_0 = (p'_{0<}$, $p'_{0>}$, $p'_{00})$, and $q'_0 = (q'_{0<}, q'_{0>}, q'_{00})$ by

$$p'_{0<} \equiv (-2E)^{1/2}\cos(t), \qquad q'_{0<} \equiv (-2E)^{1/2}\sin(t), \quad E < 0, \tag{6a}$$

$$p'_{0>} \equiv -(2E)^{1/2}\cosh(t), \qquad q'_{0>} \equiv (2E)^{1/2}\sinh(t), \quad E > 0. \tag{6b}$$

$$p'_{00} \equiv -E, \qquad q'_{00} = t, \quad 0_- \leqslant E \leqslant 0_+. \tag{6c}$$

The transformation (6a) is due to Poincaré [19]. For or all real values of E and $|t| < 2\pi$, the transformation $(-E, t) \to (p'_0, q'_0)$ is a 1:1 map from the full space of (E, t) to that of (p'_0, q'_0) and is C^2. When $t \to 0$, $q'_0 \to 0$. If one denotes negative values of E by $E_<$ and positive values of E by $E_>$, then when $t \to 0$,

$$p'_{0<} \to (-2E_<)^{1/2} \equiv p_{0<}, \qquad \text{and} \qquad E_< = -p_{0<}^2/2, \tag{7a}$$

$$p'_{0>} \to -(2E)^{1/2} \equiv p_{0>} \qquad \text{and} \qquad E_> = p_{0>}^2/2, \tag{7b}$$

while

$$0_- \leqslant p'_{00} \leqslant 0_+. \tag{7c}$$

When E is negative, $\mathbf{a}(E)$ is given by

$$\mathbf{a}(E_<) = \left(p^2 - p_{0<}^2\right)\mathbf{r}/2 - (\mathbf{r}\cdot\mathbf{p})\mathbf{p}, \tag{8a}$$

Its moving counterpart, $\mathbf{a}(-E)$, is

$$\mathbf{a}(-E_<) = \left(p^2 + p_{0<}^2\right)\mathbf{r}/2 - (\mathbf{r}\cdot\mathbf{p})\mathbf{p}. \tag{8b}$$

For $E > 0$, $\mathbf{a}(E)$ and its moving counterpart are, respectively,

$$\mathbf{a}(E_>) = \left(p^2 + p_{0>}^2\right)\mathbf{r}/2 - (\mathbf{r}\cdot\mathbf{p})\mathbf{p}, \tag{8c}$$

$$\mathbf{a}(-E_>) = \left(p^2 - p_{0>}^2\right)\mathbf{r}/2 - (\mathbf{r}\cdot\mathbf{p})\mathbf{p}. \tag{8d}$$

In the following section it will be shown that with this change of coordinates in $(-E, t)$ space, the components of $\mathbf{a}(E)$ and its moving counterpart are closely related to Cartesian momentum coordinates of relativistic free particles. This relationship connects ISO(3,1) invariant manifolds in PQET space to classical Keplerian manifolds $M(E)$ in PQE space. Thereafter, the geometric symmetries of these manifolds in the phase spaces PQ, PQE and PQET will be 'displayed', and the manifolds themselves will be further characterized. The reason these geometrical symmetries in phase space persist as dynamical symmetries in quantum mechanics will then be established.

3. The relativistic canonical transformation in PQET space

In 1909 Cunningham and Bateman discovered that solutions of Maxwell's equations can be interconverted by the spacetime inversions [14]

$$\mathbf{q}^4 \to = k^2\mathbf{q}^4/\left(q_1^2 + q_2^2 + q_3^2 - q_4^2\right), \qquad \left(q_1^2 + q_2^2 + q_3^2 - q_4^2\right) \neq 0$$

with

$$\mathbf{q}^4 = (q_1, q_2, q_3, q_4), \qquad q_4 = ct^{\text{rel}}, \tag{9a}$$

where k is a real constant, and q_1, q_2, q_3, are local Cartesian coordinates of a point in spacetime. In the phase space corresponding to the space of $\mathbf{q}^4$,

$$p^4 = (p_1, p_2, p_3, p_4), \qquad p_4 = -E^{\text{rel}}/c, \tag{9b}$$

and the Poisson bracket of two functions F and G is defined by

$$\{F, G\} = \sum_1^4 (\partial F/\partial q_j\ \partial G/\partial p_j - \partial F/\partial p_j \partial G/\partial q_j), \tag{9c}$$

Here, t^{rel} is a relativistic time, and E^{rel} is a relativistic energy equal to mc^2, with m a dynamic mass. This Poisson bracket is formally related to the classical one of (4a) by the substitution $\mathrm{E} \to E^{\text{rel}}$, $\mathrm{t} \to t^{\text{rel}}$, followed by the CT of dilatation that converts $-E^{\text{rel}}$ to $-E^{\text{rel}}/c$, and t^{rel} to ct^{rel}.

For $k^2 = 2$, the canonical transformation

$$\mathbf{q}^4 \to \mathbf{p}^4, \qquad \mathbf{p}^4 \to -\mathbf{q}^4, \tag{10a}$$

converts the spacetime inversion, (9a), to the momentum–energy inversion

$$\mathbf{p}^4 \to \mathbf{P}^4 = 2\,\mathbf{p}^4/\left(p^2 - p_4^2\right),$$

with

$$\mathbf{P}^4 = (\mathbf{P}, P_4), \qquad \mathbf{P} = (P_1, P_2, P_3), \qquad \text{and}$$
$$\mathbf{p}^4 = (\mathbf{p}, p_4), \qquad \mathbf{p} = (p_1, p_2, p_3). \tag{10b}$$

The transformation canonically conjugate to (10*b*) is

$$-\mathbf{p}^4 \to \mathbf{q}^4 \to \mathbf{Q}^4 = (\mathbf{Q}, \mathrm{Q}_4), \qquad \mathbf{Q} = (\mathrm{Q}_1, \mathrm{Q}_2, \mathrm{Q}_3),$$

with

$$\mathbf{Q} = \left(p^2 - p_4^2\right)\mathbf{q}/2 - \left(\mathbf{q}^4 \cdot \mathbf{p}^4\right)\mathbf{p}, \qquad Q_4 = \left(p^2 - p_4^2\right)q_4/2 + (\mathbf{q}^4 \cdot \mathbf{p}^4)p_4. \tag{10c}$$

The CT defined by equations (10*b*,10*c*) is a relativistic analog of a Levi Civita transformation [21]. Its inverse may be obtained by interchanging the role of the upper case and lower case letters in these equations. In regions where $\mathbf{q}^4$ and $\mathbf{p}^4$ are nonzero the transformations are smooth diffeomorphisms. One also finds that

$$-\mathbf{Q}^4 \cdot \mathbf{P}^4 = q_1 p_1 + q_2 p_2 + q_3 p_3 + q_4 p_4 = \mathbf{q} \cdot \mathbf{p} + q_4 p_4, \tag{10d}$$
$$L_{ik} = (q_i p_k - q_k p_i) = Q_i^4 P_k^4 - Q_k^4 P_i^4, \tag{10e}$$
$$K_{i4} = q_i p_4 + q_4 p_i = Q_i^4 P_4^4 + Q_4^4 P_i^4. \tag{10f}$$

Lorentz transformations in PQET phase space are symplectic transformations generated by the $\{L_{jk}\cdot\}$ and $\{K_{i4}\cdot\}$. In the Q^4, P^4 coordinate system one has

$$\{L_{ik}\cdot\} = -((Q_i\partial/\partial Q_k - Q_k\partial/\partial Q_i) + (P_i\partial/\partial P_k - P_k\partial/\partial P_i)), \tag{11a}$$
$$\{K_{i4}\cdot\} = -((Q_i\partial/\partial Q_4 + Q_4\partial/\partial Q_i) - (P_i\partial/\partial P_4 + P_4\partial/\partial P_i)). \tag{11b}$$

The three functions G

$$-\mathbf{Q}^4 \cdot \mathbf{P}^4 = \mathbf{q}^4 \cdot \mathbf{p}^4 = q_1 p_1 + q_2 p_2 + q_3 p_3 + q_4 p_4, \tag{12a}$$
$$P^2 - P_4^2 = 4\Big/\left(p^2 - p_4^2\right), \tag{12b}$$

and

$$Q^2 - Q_4^2 = (1/4)\left(q^2 - q_4^2\right)\left(p^2 - p_4^2\right)^2, \tag{12c}$$

all satisfy

$$\{L_{ij}\cdot\}G = 0, \qquad \{K_{i4}\cdot\}G = 0. \tag{12d}$$

They can therefore define Lorentz invariant manifolds in PQET space when they, or functions of them, are set equal to constants. The functions of (12*c*) arise when the transformation (10) is applied to the function $p^2 - p_4^2$, which is Poincaré invariant as well as Lorentz invariant. This Poincaré ISO(3,1) invariance becomes a more general ISO(3,1) invariance of (12*c*), with group generators $\{L_{ij}\cdot\}$, $\{K_{i4}\cdot\}$ and $\{Q_j\cdot\} = \partial/\partial P_l$, all of which annihilate $Q^2 - Q_4^2$. Taken together, the fifteen operators

$$\{L_{ij}\cdot\}, \qquad \{K_{i4}\cdot\}, \qquad \{Q_j\cdot\}, \qquad \{q_j\cdot\}, \qquad \text{and} \qquad \{Q^4 \cdot P^4\cdot\}, \tag{13}$$

satisfy the commutation relations of the Lie algebra of the conformal group obtained from the Bateman–Cunningham conformal group [16] by means of the transformation, (10*a*), that interchanges the role of positions and momenta.

4. Transformation of the inverted coordinates to their classical limit in PQE space

In the large c, classical, limit, if t is finite, then ct becomes infinite as $-E^{\mathrm{rel}}/c \to 0$. To transform the coordinates and relations of the previous section to corresponding non-relativistic coordinates and relations in PQE space, we therefore choose a reference frame in which $t = 0$ when $\mathrm{t}^{\mathrm{rel}} = 0$. We then set $E^{\mathrm{rel}} = E_0 \pm |\Delta E|$, where $\pm\,|\Delta E|$ is the classical energy of the system, $E_0 = m_0c^2$ being the mass energy it would have in a rest frame in which no forces are acting. One has

$$p_4^2 = (-E^{\mathrm{rel}}/c)^2 = (-(E_0 \pm |\Delta E|)/c)^2 = \left(E_0^2 + |\Delta E|^2\right)/c^2 \pm 2m_0|\Delta E|. \qquad (14a)$$

In the large c limit,

$$p_4^2 \to \pm 2m_0|\Delta E|. \qquad (14b)$$

Assuming m_0 has unit value, and setting $2\,|\Delta E| = p_0^2$, one has

$$p_4^2 \to \pm p_0^2. \qquad (14c)$$

In light of (7a), (7b), for nonzero $|\Delta E|$, this gives rise to the four cases

$$p_4^2 \to p_{0<}^2 = -2E_<, \qquad (14d)$$

$$p_4^2 \to p_{0>}^2 = 2E_>, \qquad (14e)$$

and

$$p_4^2 \to -p_{0<}^2 = 2E_<, \qquad (14f)$$

$$p_4^2 \to -p_{0>}^2 = -2E_>. \qquad (14g)$$

The previous equations do not imply any relation between the sign of p_4 and that of p_0. However, relations (6) of section 2, require that if

$$p_4^2 \to \pm p_{0<}^2, \qquad \text{then} \quad p_4 \to p_{0<}, \qquad (14h)$$

and, that if E is not negative, either

$$p_4 \to -p_{0>}, \qquad \text{or} \qquad p_4 \to -p_{00}. \qquad (14i)$$

Using these limits, one obtains transformations from the $\mathbf{P}^4$, $\mathbf{Q}^4$ coordinates in PQET space to corresponding classical coordinates $\mathbf{P}^{\mathrm{c}}$, $\mathbf{Q}^{\mathrm{c}}$, in PQE space. When $p_4^2 \to \pm p_0^2$,

$$\mathbf{P} \to \mathbf{P}^{\mathrm{c}}\left(\pm p_0^2\right) = 2\mathbf{p}\big/\left(p^2 - \pm p_0^2\right), \qquad (15a)$$

The components of $\mathbf{P}^{\mathrm{c}}\left(-p_{0<}^2\right)$ are $2p_j\big/\left(p^2 + p_{0<}^2\right)$, which are proportional to three of the projective momentum variables Fock introduced in his discussion of the degeneracy of the discrete energy levels of hydrogen-like atoms [1]. Applying the limits to the Q's one has

$$\mathbf{Q} \to \mathbf{Q}^{\mathrm{c}}\left(\pm p_0^2\right) = \left(p^2 - \pm p_0^2\right)\mathbf{q}/2 - (\mathbf{q}\cdot\mathbf{p})\mathbf{p}. \qquad (15b)$$

Using (14) the $\mathrm{Q}^c\left(\pm\, p_0^2\right)$ may be converted to functions of E. In particular,

$$\mathbf{Q}^{\mathrm{c}}\left(p_{0<}^2\right) = \left(p^2 - p_{0<}^2\right)\mathbf{q}/2 - (\mathbf{q}\cdot\mathbf{p})\mathbf{p} = \mathbf{a}(E_<), \qquad (15c)$$

and

$$\mathbf{Q}^{\mathrm{c}}\left(-p_{0>}^2\right) = \left(p^2 + p_{0>}^2\right)\mathbf{q}/2 - (\mathbf{q}\cdot\mathbf{p})\mathbf{p} = \mathbf{a}(E_>). \qquad (15d)$$

These stationary vectors $\mathbf{a}(E)$, are converted to their moving counterparts $\mathbf{a}(-E)$ by changing the signs before p_0^2 in the corresponding $\mathbf{Q}^{\mathrm{c}}$.

Applying the limits to $P_4 = 2p_4/\left(p^2 - p_4^2\right)$, one obtains

$$P_4^{\mathrm{c}}\left(\pm p_{0<}^2\right) = 2p_{0<}/\left(p^2 - \pm p_{0<}^2\right), \tag{15e}$$

and

$$P_4^{\mathrm{c}}\left(\pm p_{0>}^2\right) = -2p_{0>}/\left(p^2 - \pm p_{0<}^2\right). \tag{15f}$$

From $Q_4 = (\mathbf{q}\cdot\mathbf{p})p_4$, one obtains

$$Q_4^{\mathrm{c}}\left(\pm p_{0<}^2\right) = (\mathbf{q}\cdot\mathbf{p})p_{0<} \equiv a_4(E_<), \tag{15g}$$

and

$$Q_4^{\mathrm{c}}\left(\pm p_{0>}^2\right) = -(\mathbf{q}\cdot\mathbf{p})p_{0>} \equiv a_4(E_>). \tag{15h}$$

Substituting relations (15) into (12b), (12c) yields the transformation

$$P^2 - P_4^2 \to 4/\left(p^2 - \pm p_0^2\right), \tag{16a}$$

and

$$Q^2 - Q_4^2 \to (1/4)(q^2)\left(p^2 - \pm p_0^2\right)^2. \tag{16b}$$

For $\pm p_0^2 = 2E$, the latter equation implies

$$((1/4)(q^2)(p^2 - 2E)^2)^{1/2} = H. \tag{16c}$$

Finally, it should be noted that

$$\mathbf{Q}^4\cdot\mathbf{P}^4 \to \mathbf{Q}^{\mathrm{c}}\left(\pm p_0^2\right)\cdot\mathbf{P}^{\mathrm{c}}\left(\pm p_{0>}^2\right) = -(\mathbf{q}\cdot\mathbf{p}). \tag{16d}$$

At perhelion, where $\mathrm{t} = t^{\mathrm{rel}} = 0$ and $\mathbf{q}\cdot\mathbf{p} = 0$, this implies that the relativistic position and momentum 4-vectors, $\mathbf{Q}^4$, $\mathbf{P}^4$ are orthogonal, as are the classical position and momentum 4-vectors, $\mathbf{Q}^{\mathrm{c}}$ and $\mathbf{P}^{\mathrm{c}}$.

To summarize: the transformation from the initial Cartesian 4-vectors $\mathbf{q}$, $\mathbf{p}$ to the relativistic 4-vectors $\mathbf{Q}^4$, $\mathbf{P}^4$ to the non-relativistic 4-vectors $\mathbf{Q}^{\mathrm{c}}$, $\mathbf{P}^{\mathrm{c}}$ has the following effects:

(a) It converts free-particle momentum vectors $(\mathbf{p}, -E^{\mathrm{rel}}/c)$ into 4-vectors $\mathbf{Q}^4$, and these into $\mathbf{Q}^{\mathrm{c}}$ whose first three components are the three Runge–Lenz constants of motion, $a_{\mathrm{j}}(E)$, and whose fourth component, $a_4(E)$, is proportional to $\mathbf{q}\cdot\mathbf{p}$. It also creates the moving counterparts of these functions and vectors.

(b) It produces a 4-vector $\mathbf{P}^4$, conjugate to $\mathbf{Q}^4$, and then converts $\mathbf{P}^4$ to $\mathbf{P}^{\mathrm{c}}$, whose first three components are proportional to three of Fock's projective momenta. (The relation of its fourth component to Fock's fourth projective momentum component is discussed in section 5.)

(c) It transforms the Poincaré-invariant free-particle Hamiltonian system defined in PQE space by

$$H_{fp}^{\mathrm{rel}} \equiv \left(\mathbf{p}^2 - p_4^2\right)^{1/2} = K, \tag{17a}$$

into the Lorentz and ISO(3,1) invariant Hamiltonian system defined by

$$H_K^{\mathrm{rel}} \equiv \left((Q^4)^2 - \left(Q_4^4\right)^2\right)^{1/2} = (1/2)\left(q^2 - q_4^2\right)^{1/2}\left(p^2 - p_4^2\right) = K. \tag{17b}$$

It then converts this manifold into the regularized classical Kepler Hamiltonian system defined in PQE space by

$$H = (1/2)(q^2)^{1/2}(p^2 - 2E) = K. \tag{17c}$$

The Lie generators of the invariance groups of equations (17a)–(17c) are PB operators, $\{F_j\,\cdot\}$, that satisfy $\{F_j\,\cdot\}H\,| = 0$. That is, $\{F_j\,\cdot\}H = 0$, when $H = \mathrm{K}$. The transformations carried out by $\exp(a_j\{F_j \cdot\})$, with group parameter α, leave the equations of motion invariant, and convert solutions of the Hamiltonian equations of motion of H into solutions. The commutation relations of the invariance generators determine local dynamical symmetry groups, and it is the action of these generators on the coordinates of points in phase space that expresses the symmetry of manifolds in PQE and PQET space. The degeneracy groups of the systems are defined by operators $\{F_j \cdot\}$ that do not alter the energy of the system. They define submanifolds $M(E)$ in six-dimensional PQ phase space. Equations (17*a*)–(17*c*) do not, as they stand, fully define these manifolds. The additional functions required for the classical Kepler system are obtained in the following section.

5. Geometrical interpretation of the classical degeneracy groups in phase space

In this section, the results of the previous sections are used to extend the moving three vectors $\mathbf{a}(-E)$, of section 1, to corresponding 4-vectors $\boldsymbol{\rho}(E)$ and $\boldsymbol{\pi}(E)$, in PQE space. To accomplish this, the action of the PB operators $\{A_j(E)\cdot\}$ upon $\mathbf{Q}^c$ and $\mathbf{P}^c$ is examined. The moving vectors $\boldsymbol{\rho}(E)$ become identified with the $\mathbf{Q}^c(-E)$. The action of $\{A_j(E)\cdot\}$ on the $\mathbf{P}^c$ directly determines the components of 4-vectors, $\boldsymbol{\pi}(E)$, that have the same transformation properties as the $\boldsymbol{\rho}(E)$. The fourth components of the $\boldsymbol{\pi}(E)$ are shown not to be identical to those of the $\mathbf{P}^c$. The generators of the degeneracy groups become first-order differential operators of the form (4*d*), (4*e*), but with q's and p's replaced by $\boldsymbol{\rho}$'s and $\boldsymbol{\pi}$'s. The regularized Hamiltonian functions $H(E)$, and two further quadratic functions of the $\boldsymbol{\rho}(E)$ and $\boldsymbol{\pi}(E)$, all with the symmetry of $H(E)$, will be shown to determine the manifolds $M(E)$ upon which points representing Kepler motions evolve in phase space.

We first consider bound-state Kepler motion with $E = E_< = -p_{0<}^2\big/2$. One may define four locally Cartesian coordinates of a point at the terminus of the moving vector $\boldsymbol{\rho}_< = (\mathbf{a}(-E_<), \mathbf{a_4}(-E_<))$ by

$$\rho_{j<} \equiv \left(p^2 + p_{0<}^2\right) q_j/2 - (\mathbf{q}\cdot\mathbf{p})p_j, \qquad j = 1, 2, 3, \tag{18a}$$

and

$$\rho_{4<} \equiv (\mathbf{q}\cdot\mathbf{p})p_{0<}. \tag{18b}$$

The vector $\boldsymbol{\rho}_<$ is a special case of the vector $\mathbf{Q}^c$. One finds that

$$\rho_<^2 \equiv \sum_1^4 \rho_{j<}^2 = ((r/2)\left(p^2 + p_{0<}^2\right)^2 = H(E_<)^2, \tag{19}$$

It follows that for $E < 0$, the defining equation $H = \mathrm{K}$, may be expressed as $\rho_< = K$.

The 'renormalized' stationary three-vector, $\mathbf{A}_<$ has components

$$A_{j<} = (1/p_{0<})\left(\left(p^2 - p_{0<}^2\right)q_j/2 - (\mathbf{q}\cdot\mathbf{p})p_j\right), \tag{20}$$

The PB operators $\{A_{i<} \cdot\}$ and $\{L_{ij} \cdot\}$ obey the SO(4) commutation relations:

$$[\{A_{i<}\cdot\}, \{A_{j<}\cdot\}] = \{L_{ij}\cdot\}, \qquad [\{L_{ij}\cdot\}, \{A_{j<}\cdot\}] = -\{A_{i<}\}, \qquad \text{and}$$
$$[\{L_{ij}\cdot\}, \{L_{jk}\cdot\}] = -\{L_{ik}\cdot\}. \tag{21}$$

The SO(4) group operators $\exp(\alpha_j\{A_{j<}\})$ and $\exp(\lambda_{ij}\{L_{ij}\})$ move points at the terminus of $\boldsymbol{\rho}_< = (\rho_{1<}, \rho_{2<}, \rho_{3<}, \rho_{4<})$. The action of $\{A_{i<}\cdot\}$ and $\{L_{ij}\cdot\}$ upon the $\rho_{j<}$ is as follows:

$$\begin{aligned} &\{A_{i<}\cdot\}\rho_{j<} = \delta_{ij}\,\rho_{4<}, \qquad \{A_{i<}\cdot\}\rho_{4<} = -\rho_{i<}, \\ &\{L_{ij}\cdot\}\rho_{j<} = -\rho_{i<} \end{aligned} \tag{22}$$

Thus, both the $\{A_{i<}\cdot\}$ and the $\{L_{ij}\cdot\}$ generate rotations of the 4-vector $\rho_<$. These rotations move points on a surface with

$$\boldsymbol{\rho}_< \cdot \boldsymbol{\rho}_< = (r(p^2 - 2E)/2)^2 = H = \mathrm{K}^2, \tag{23}$$

which therefore has SO(4) rotational symmetry. Consequently, for $E < 0$, the equation $H = K$ defines an $S(3)$ hypersphere with radius K.

Next let

$$\pi_{i<} = 2p_1 \big/ \left(p^2 + p_{0<}^2\right), \qquad i = 1, 2, 3. \tag{24a}$$

The action of the $\{A_{i<}\cdot\}$ on these produces a further momentum coordinate,

$$\pi_{4<} \equiv (1/p_{0<}) \left(p^2 - p_{0<}^2\right) \big/ \left(p^2 + p_{0<}^2\right). \tag{24b}$$

It and the other π's satisfy the relations

$$\{A_{i<}\cdot\}\pi_{j<} = \delta_{ij}\pi_{4<}, \qquad \{A_{i<}\cdot\}\pi_{4<} = -\pi_{i<}. \tag{24c}$$

The four functions $\pi_{i<}$ are Fock's projective momentum coordinates [1], divided by $p_{0<}$. They satisfy the identity

$$E_< = (-1/2) \big/ \left(\pi_{1<}^2 + \pi_{2<}^2 + \pi_{3<}^2 + \pi_{4<}^2\right). \tag{25}$$

Equations (24) imply that $\{A_{i<}\cdot\}$ generate rotations of the locally Cartesian 4-vectors $\boldsymbol{\pi}_< \equiv (\pi_{1<}, \pi_{2<}, \pi_{3<}, \pi_{4<})$. The $\{L_{ij}\cdot\}$ also generate rotations of the $\boldsymbol{\pi}_<$, and equations (22)–(25) imply that in six-dimensional PQ space, rotations on the two hyperspheres defined by (23) and (25), are carried out by transformations generated by Poisson-bracket operators which have the realization

$$\begin{aligned}
&\{A_{i<}\cdot\} = -(\rho_{1<}\partial/\partial\rho_{4<} - \rho_{4<}\partial/\partial\rho_{i<} + \pi_{1<}\partial/\partial\pi_{4<} - \pi_{4<}\partial/\partial\pi_{1<}), \qquad i = 1, 2, 3,\\
&\{L_{ij}\cdot\} = -(\rho_{i<}\partial/\partial\rho_{j<} - \rho_{j<}\partial/\partial\rho_{i<} + \pi_{i<}\partial/\partial\pi_{j<} - \pi_{j<}\partial/\partial\pi_{1<}), \qquad i < j = 1, 2, 3.
\end{aligned} \tag{26}$$

The evolution of $\boldsymbol{\rho}_<$ is determined by the equation

$$\mathrm{d}\boldsymbol{\rho}_</\mathrm{d}\tau = \{\boldsymbol{\rho}_<, \rho_<\} = p_{0<}^2 H\, \boldsymbol{\pi}_< = p_{0<}^2 K\, \boldsymbol{\pi}_<. \tag{27a}$$

and that of the $\boldsymbol{\pi}_<$ is determined by

$$\mathrm{d}\boldsymbol{\pi}_</\mathrm{d}\boldsymbol{\tau} = \{\boldsymbol{\pi}_<, \rho_<\} = -\boldsymbol{\rho}_</H = -\boldsymbol{\rho}_</K. \tag{27b}$$

Consequently,

$$\mathrm{d}^2\boldsymbol{\rho}_</\mathrm{d}\boldsymbol{\tau}^2 = -p_{0<}^2\boldsymbol{\rho}_<, \qquad \mathrm{d}^2\boldsymbol{\pi}_</\mathrm{d}\tau^2 = -p_{0<}^2\boldsymbol{\pi}_<. \tag{28}$$

The motions of $\boldsymbol{\rho}_<$ and $\boldsymbol{\pi}_<$ are constrained by the further relation

$$\sum_1^4 \rho_{j<}\pi_{j>} \equiv \boldsymbol{\rho}_< \cdot \boldsymbol{\pi}_< = 0. \tag{29}$$

Thus, for each value of E, the bound-state motions in six-dimensional PQ phase space evolve on an SO(4) invariant manifold $M(E)$, defined by the three equations

$$\boldsymbol{\rho}_< \cdot \boldsymbol{\rho}_< = K^2, \tag{30a}$$

$$(p_{0<}\pi_<) \cdot (p_{0<}\pi_<) = 1, \tag{30b}$$

$$\boldsymbol{\rho}_< \cdot \boldsymbol{\pi}_< = 0. \tag{30c}$$

The two vectors $\boldsymbol{\rho}_<$ and $p_{0<}\boldsymbol{\pi}_<$ may be considered to originate at the origin $\mathbf{p} = 0$, $\mathbf{q} = 0$ in phase space, and, because of (30c), the vectors $\boldsymbol{\rho}_< + p_{0<}\boldsymbol{\pi}_<$ and $\boldsymbol{\rho}_< - p_{0<}\boldsymbol{\pi}_<$ both terminate on the surface of an S(3) hypersphere of radius $(K^2+1)^{1/2}$. As the system evolves, the angle between them, $\theta = \arccos((K^2-1)/(K^2+1))$, remains constant.

We conclude this section with a brief discussion of Keplerian systems with fixed non-negative energies. When the energy is positive, $E = E_> = p_{0>}^2\big/2$. The constant reference 3-vector, $\mathbf{A}_> \equiv \mathbf{a}_>/|p_{0>}|$ has components

$$A_{j>} = \left(\left(p^2 + p_{0>}^2\right) q_j/2 - (\mathbf{r}\cdot\mathbf{p})p_j\right)/ip_{0>}i, \tag{31}$$

and the moving 4-vector $\boldsymbol{\rho}_>$ has components

$$\boldsymbol{\rho}_{j>} = \left(p^2 - p_{0>}^2\right) q_j/2 - (\mathbf{r}\cdot\mathbf{p})p_j, \qquad i = 1, 2, 3; \tag{32a}$$

$$\boldsymbol{\rho}_{4>} \equiv -(\mathbf{r}\cdot\mathbf{p})p_{0>}. \tag{32b}$$

The corresponding components of the vector $\boldsymbol{\pi}_>$ are

$$\pi_{j>} = 2p_j/(p^2 - p_{0>}), \qquad \pi_{4>} \equiv (1/p_{0>})\left(p^2 + p_{0>}^2\right)\Big/\left(p^2 - p_{0<}^2\right). \tag{33}$$

One finds

$$\boldsymbol{\rho}_{1>}^2 + \boldsymbol{\rho}_{2>}^2 + \rho_{3>}^2 - \rho_{4>}^2 = H^2 \tag{34a}$$

$$E_> = (1/2)\Big/\left(\pi_{4>}^2 - \left(\pi_{1>}^2 + \pi_{2>}^2 = \pi_{3>}^2\right)\right), \tag{34b}$$

$$\sum_1^3 \rho_{j>}\pi_{j>} + \rho_{4>}\pi_{4>} = 0. \tag{34c}$$

The action of the PB operators of $A_{i>}$ and L_{ij} on the coordinates $\rho_{j>}$, and $\pi_{j>}$ is expressed by

$$\begin{aligned} \{A_{i>}\cdot\} &= -(\rho_{i>}\partial/\partial\rho_{4>} + \rho_{4>}\partial/\partial\rho_{i>}) + \pi_{i>}\partial/\partial\pi_{4>} + \pi_{4>}\partial/\partial\pi_{i>} \qquad i = 1, 2, 3, \\ \{L_{ij}\cdot\} &= -(\rho_{i>}\partial/\partial\rho_{j>} - \rho_{j>}\partial/\partial\rho_{i>} + \pi_{i>}\partial/\partial\pi_{j>} - \pi_{j>}\partial/\partial\pi_{i>}), \qquad i < j = 1, 2, 3. \end{aligned} \tag{35}$$

These generate an SO(3,1) invariance group of relations (34*a*–*c*). The evolution of $\boldsymbol{\rho}_>$ and $\boldsymbol{\pi}_>$, is governed by the equations

$$\mathrm{d}\boldsymbol{\rho}_>/\mathrm{d}\tau = \{\boldsymbol{\rho}_>, \rho_>\} = -2E_> H\boldsymbol{\pi}_>, \tag{36a}$$

$$\mathrm{d}/\mathrm{d}\tau = \{\boldsymbol{\pi}_>, \rho_>\} = -\boldsymbol{\rho}_>/H \tag{36b}$$

and

$$\mathrm{d}^2\boldsymbol{\rho}_>/\mathrm{d}\tau^2 = p_{0>}^2\boldsymbol{\rho}_>, \tag{36c}$$

$$\mathrm{d}^2\boldsymbol{\pi}_>/\mathrm{d}t^2 = p_{0>}^2\boldsymbol{\pi}_>. \tag{36d}$$

Equations (34) determine an SO(3,1) invariant 3-manifold in phase space upon which the termini of the vectors $\boldsymbol{\rho}_> + p_{0>}\boldsymbol{\pi}_{<>}$ and $\boldsymbol{\rho}_{<>} - p_{0>}\boldsymbol{\pi}_>$ evolve.

When $E = 0$, one has

$$a_{j0} = p^2 q_j/2 - (\mathbf{r}\cdot\mathbf{p})p_j, \qquad j = 1, 2, 3; \tag{37a}$$

$$a_{40} = (\mathbf{r}\cdot\mathbf{p})p_{00}. \tag{37b}$$

Consequently,

$$\rho_{j0} = a_{j0}, \tag{37c}$$

$$\rho_{40} = (\mathbf{r}\cdot\mathbf{p})p_{00}, \tag{37d}$$

with $0_- \leqslant p'_{00} \leqslant 0_+$. The components of the vector $\boldsymbol{\pi}_0$ are

$$\pi_{j0} = 2p_j/p^2, \qquad j = 1, 2, 3. \tag{38a}$$

while

$$p_{00}\pi_{40} = 1. \tag{38b}$$

It follows that

$$\sum_1^4 \rho_{j0}^2 = (rp^2/2)^2 = H^2, \tag{39a}$$

$$\sum_1^4 \pi_{j0}^2 = 4/p^2, \tag{39b}$$

$$\sum_1^4 \rho_{j0}\pi_{j0} = 0. \tag{39c}$$

One finds

$$\{a_{j0}\cdot\} = \partial/\partial\pi_{j0}, \qquad j = 1, 2, 3, \tag{39d}$$

$$\{L_{ij}\cdot\} = -(\rho_{i0}\partial/\partial\rho_{j0} - \rho_{j0}\partial/\partial\rho_{i0} + \pi_{i0}\partial/\partial\pi_{j0} - \pi_{j0}\partial/\partial\pi_{i0}), \qquad i < j = 1, 2, 3. \tag{39e}$$

Because $\mathbf{a}(0) = \mathbf{a}(E) + \mathbf{a}(-E)$, a Lie algebra that contains $\mathbf{a}(E)$ and $\mathbf{a}(-E)$ also contains $\mathbf{a}(0)$. The evolution of the vectors $\boldsymbol{\rho}_0$ and $\boldsymbol{\pi}_0$ is determined by the equations

$$\mathrm{d}\boldsymbol{\rho}_0/\mathrm{d}\tau = \{\boldsymbol{\rho}_0, \boldsymbol{\rho}_0\} = 0, \tag{40a}$$

$$\mathrm{d}\boldsymbol{\pi}_0/\mathrm{d}\tau = \{\pi_0, \rho_0\} = -\boldsymbol{\rho}_0/H. \tag{40b}$$

Consequently,

$$\mathrm{d}^2\boldsymbol{\rho}_0/\mathrm{d}\tau^2 = 0, \tag{40c}$$

$$\mathrm{d}^2\boldsymbol{\pi}_{0,}/\mathrm{d}t^2 = 0. \tag{40d}$$

To summarize: as E varies, $\mathbf{a}(E)$, $\boldsymbol{\rho}(E) = \mathbf{a}(-E)$, and $\boldsymbol{\pi}(E)$, all adjust themselves in such a manner that the relation $r(H - E) = 0$ and the equations of motion, together determine manifolds in PQET phase space defined by three relations

$$\left(\rho_1^2 + \rho_2^2 + \rho_3^2 + \mathrm{g}^{44}\rho_4^2\right)^{1/2} = K, \tag{41a}$$

$$E_> = (-1/2)\Big/\left(\pi_1^2 + \pi_2^2 + \pi_3^2 + g_{44}\pi_4^2\right), \tag{41b}$$

$$\left(\rho_1\pi_1 + \rho_2\pi_2 + \rho_3\pi_3 + \mathrm{g}_4^4\,\rho_4\pi_4\right) = 0, \tag{41c}$$

with

$$g^{44} = g_{44} = g_4^4 = 1, \qquad \text{if} \quad E < 0; \tag{41d}$$

$$g^{44} = g_{44} = 0, \qquad g_4^4 = 1, \qquad \text{if} \quad E = 0 \tag{41e}$$

and

$$g^{44} = g_{44} = -1, \qquad g_4^4 = 1, \qquad \text{if} \quad E > 0. \tag{41f}$$

For each range of E, the points that represent the systems evolve on manifolds in phase determined by these equations. The manifolds have geometric symmetries determined by the Lie generators of the corresponding degeneracy group, all of which are first-order differential operators that are of degree one or zero in the variables ρ_j and π_j. For any given value of E the relations in (41) constrain the motions to curved three-dimensional manifolds in six-dimensional PQ space. These together with the evolution equations (28), (36), (40) display in a simple manner the general relationships between Kepler motions in phase space and equivalent flows on manifolds of constant curvature relationships established in the previously mentioned papers of Moser, Souriau, Belbruno and Osipov [22–25].

6. The SO(4,1) and SO(4,2) dynamical groups

In clarifying the geometric interpretation of the SO(4,1) and SO(4,2) transformations that change E in PQE space it is helpful to establish a connection with the treatment of the regularized hydrogen-like given by Barut and Bornzin [35] and Bednar [36]. The operators J_{ab} and K_{ab}, of the classical treatment given here are in 1:1 correspondence with the Schrödinger operators used by Bednar in his treatment of the regularized Schrödinger equation. (Because the Hamiltonian operator is multiplied by r, the scalar products used to define operator adjointness are not the usual ones. However, they, and the form of the self-adjoint group generators, are well known: the momentum-space scalar product is that of Fock, and from this it follows that one must insert a factor of $1/r = \left(p^2 + p_{0<}^2\right)/2p_{0<}^2$ into the position-space scalar product of Schrödinger [37]).

Define the PB operators

$$S(\alpha) = \exp(\alpha D), \tag{42a}$$

$$D = \{(\mathbf{q}\cdot\mathbf{p})\cdot\} = \mathbf{p}\nabla_{\mathrm{p}} - \mathbf{q}\cdot\nabla_{\mathrm{q}}. \tag{42b}$$

Then

$$S(\alpha)\mathbf{p} = \exp(\alpha)\mathbf{p}, \tag{42c}$$

$$S(\alpha)\mathbf{q} = \exp(-\alpha)\mathbf{q}. \tag{42d}$$

These finite transformations may be used to convert the vectors $\mathbf{a}(E)$ with $E = \pm 1/2$ to vectors $\mathbf{A}(E)$. If $\exp(\alpha) = p_0$, then,

$$S(\alpha)\mathbf{a}(-1/2) = S(\alpha)\left(((p^2-1)\mathbf{q}/2 - \mathbf{q}\cdot\mathbf{p}\,\mathbf{p}) - \mathbf{q}/2\right) = \mathbf{A}\left(-p_0^2/2\right), \tag{42e}$$

and

$$S(\alpha)\mathbf{a}(1/2) = \mathbf{A}\left(p_0^2/2\right). \tag{42f}$$

Because $\mathbf{a}(0) = (\mathbf{a}(-1/2) + \mathbf{a}(1/2))/2$, it is not necessary to treat the $E = 0$ case separately. The general effect of $S(\alpha)$ on $\mathbf{a}(-1/2)$ and $\mathbf{a}(1/2)$ is such that

$$S(\alpha)\{\mathbf{a}(1/2)\cdot\} = \cosh(\alpha)\,\{\mathbf{a}(1/2)\cdot\} + \sinh(\alpha)\,\{\mathbf{a}(-1/2)\cdot\}, \tag{43a}$$

and

$$S(\alpha)\{\mathbf{a}(-1/2)\cdot\} = \cosh(\alpha)\,\{\mathbf{a}(-1/2)\cdot\} + \sinh(\alpha)\,\{\mathbf{a}(1/2)\cdot\}. \tag{43b}$$

The relations (42), (43) enable one to use D and the components of $\mathbf{a}(-1/2)$ and $\mathbf{a}(1/2)$ to provide PB operators that constitute a basis for the Lie algebras and groups that act in the $E < 0$, $0_- \leqslant E \leqslant 0_+$ and $E > 0$, subspaces of PQET space. Set $D = K_{45} = K_{45}$, and for $i < j$, $j = 1, 2, 3$, define the operators

$$J_{ij} = \{L_{ij}\cdot\} = -J_{ji}, \tag{44a}$$

$$J_{j4} = \{a_j(-1/2)\cdot\} \equiv -J_{4j}, \tag{44b}$$

$$K_{j5} = \{a_j(1/2)\cdot\} \equiv K_{5j}. \tag{44c}$$

The J_{ij}, J_{j4} satisfy the commutation relations of the SO(4) bound-state degeneracy group of the Kepler system, and the J_{ij}, K_{j5} satisfy the commutation relations of the SO(3,1), $E > 0$, degeneracy group. The J_{ij}, J_{j4}, K_{j5}, and K_{45} together generate an SO(4,1) invariance group of $H(E)$ and the equations of motion. The group generated by K_{45} and these nine operators is the SO(4,1) dynamical group of Kepler systems: the three degeneracy subgroups of the ten-parameter group SO(4,1) are invariance groups of the regularized classical Kepler Hamiltonian, H(E), for different ranges of E. It corresponds to the ten-parameter invariance group ISO(3,1) of the relativistic Hamiltonians of (12*b*), (12*c*).

The SO(4,1) group is a subgroup of an SO(4,2) group which is generated by the ten operators that generate SO(4,1) and the five further operators

$$J_{56} \equiv \{-r(p^2+1)/2\cdot\} \equiv -J_{65}, \tag{45a}$$

$$K_{46} \equiv \{r(p^2-1)/2\cdot\} \equiv K_{64}, \tag{45b}$$

$$K_{j6} \equiv \{r\ p_j\cdot\} \equiv K_{6j}, \qquad j = 1, 2, 3. \tag{45c}$$

The operator J_{56} carries out rotations that are those of the angle variable of an action-angle pair, and K_{46} carries out analogous hyperbolic rotations. These two, together with K_{45}, generate an SO(2,1) transformation group. Composing rotations generated by J_{56} with hyperbolic transformations generated by the K_{i5} produces the transformations generated by the K_{j6}. The K_{j6} also arise when the rotations generated by the J_{i4} are composed with the hyperbolic rotations generated by K_{46}. The nonzero commutators of the 15 SO(4,2) generators may be put in the form

$$[J_{ab}, J_{bc}] = -J_{ac}, \qquad [J_{ab,} K_{bc}] = -K_{ac}, \qquad [K_{ab}, K_{bc}] = -J_{ac}. \tag{46}$$

7. Lie algebraic extension of Dirac's correspondence principle

In this section we establish conditions under which Dirac's correspondence principle produces an isomorphism between the Lie algebra of a closed set of Poisson-bracket operators and the Lie algebra of a corresponding set of quantum-mechanical operators.

The components $Q^4_{op\ j}$, $P^4_{op\ k}$, of quantum mechanical analogs of the canonically conjugate inversion variables (11*b*), (11*c*) can satisfy

$$\left[Q^4_{op\ j}, P^4_{op\ k}\right] = i(h/2\pi)\delta_{j,k}, \qquad \left[Q^4_{op\ j}, Q^4_{op\ k}\right] = 0 = \left[P^4_{op\ j}, P^4_{op\ k}\right]. \tag{47}$$

They are able to obey Dirac's correspondence principle without being of the functional form Dirac assumed when advancing the principle. Because of this, and because the principle itself can produce ambiguities, we begin our discussion with a brief statement of the form of the correspondence principle which we will use. Suppose the members of a set of classical mechanical functions X_j^0 are of the general form

$$X_j^0(q, t, p, E) = \sum_\nu f_{\nu j}(q, t) g_{\nu j}(p, E), \tag{48a}$$

and suppose that f's and g's are polynomial functions of their arguments. Let the corresponding self-adjoint quantum mechanical operators be

$$\chi_j^0 = (1/2) \sum_\nu (f_{\nu j}(q, t) g_{\nu j}(p_{op}, E_{op}) + (f_{\nu j}(q) g_{\nu j}(p_{op}, E_{op}))^+, \tag{48b}$$

or

$$\chi_j^0 = (1/2) \sum_\nu (f_{\nu j}(q_{op}, t_{op}) g_{\nu j}(p, E) + (f_{\nu j}(q_{op}, t_{op}) g_{\nu j}(p, E))^+. \tag{48c}$$

For a given definition of adjointness, equations (48) set up a two-way, 1:1, correspondence between quantum mechanical operators and classical functions. Though the adjoints of $E_{op} = \mathrm{i}\partial/\partial \mathrm{t}$ and $t_{op} = -\mathrm{i}\partial/\partial \mathrm{E}$ are not defined in Schrödinger mechanics, we shall suppose that they are appropriately defined whenever they are involved in correspondences of the form defined in (48). In this paper we often avoid the problem by treating E as a parameter rather than a dynamical variable. Dirac noted that his correspondence principle applied to polynomial functions of q's and p's, and then pointed out that similarity transformations in quantum mechanics take on the role played by canonical transformations in classical mechanics [38]. Thus, for Dirac's correspondence principle to be obeyed it is sufficient, but not necessary, that X_j^0 and χ_j^0 be of the form (48*b*) or (48*c*). If the quantum mechanical q's and p_{op}'s or p's and q_{op}'s of (48) undergo a similarity transformation with operator S that preserves their quantum mechanical conjugacy, then the classical conjugacy of q', and p's will also be preserved if the corresponding classical transformation, T, is a symplectic diffeomorphism. This sets up the correspondences

$$X_j^0 \leftrightarrow T\ X_j^0\ T^{-1}, \tag{49a}$$

and

$$\chi_j^0 \leftrightarrow S\ \chi_j^0\ S^{-1}, \tag{49b}$$

correspondences between functions X and operators χ that is more general than that provided via (48), for the $f_{\nu j}$ and need $g_{\nu j}$ no longer be polynomial functions of their arguments.

The variables Q_i and P_j introduced in section 3 are related to ordinary Cartesian q's and p's by a transformation which produces a singularity at the origin in PE momentum space. For this reason the argument justifying (49) is problematical. However, if for example, one replaces q_j in each Q_k by$/(h/2\pi)\partial/\partial p_j$, supposed self-adjoint, and puts the result in the symmetrized self-adjoint form given in (48*c*), one obtains the relations given in (47). Equation (47) exemplifies a two-way correspondence $X_j \leftrightarrow \chi_j$, that obeys Dirac's correspondence principle—a principle that can be valid even when the functions and operators are not of the polynomial forms given in (48) or (49). For this reason we shall henceforth only assume that the functions X_j and functionals χ_j may be of the more general symmetrized form obtained by relaxing the requirement that the $f_{\nu j}$ and $g_{\nu j}$ in (48) be polynomials. As is often the case when attempting to apply the correspondence principle, each case must be separately investigated.

Now suppose that a set of such symmetrized self-adjoint functions X_j are members of a Poisson-bracket Lie algebra of functions

$$\{X_i, X_j\} = c_{ij}^k X_k. \tag{50a}$$

If Dirac's principle is valid, it requires that the corresponding symmetrized self-adjoint operators χ_j are members of the commutator Lie algebras

$$(-\mathrm{i}/(h/2\pi))[\chi_j, \chi_j] = c_{ij}^k \chi_k. \tag{50b}$$

To determine the relation of these two Lie algebras to the commutator Lie algebras of Poisson-bracket operators $\{X_i\cdot\}$, let F be any C^2 function of p's and q's. Then, on applying the Jacobi identity to the relation

$$[\{X_i\}, \{X_j\cdot\}]F = \{X_i, \{X_j, F\}\}, \tag{50c}$$

one establishes that if X_j satisfy the PB relations (50*a*), then X_j satisfy the commutation relations

$$[X_i, X_j] = c_{ij}^k X_k, \tag{50d}$$

with identical structure constants. However, if any choice of basis in (50*a*) (or (50*b*)) has an X_m (or a χ_m) that is a constant, then the corresponding $X_m = \{X_m\cdot\}$ in (50*d*) will vanish. If the constant is not zero, the Lie algebra of Poisson-bracket operators will not then be usefully isomorphic to the quantum mechanical Lie algebra, even when Dirac's correspondence principle holds true. The oscillator energy-shift algebra of Dirac provides an example of this: it has the Heisenberg algebra q, p_s, $\mathrm{i}(h/2\pi)$, as a basis, and $\{q\cdot\} = -\partial/\partial p$, $\{p\cdot\} = -\partial/\partial q$, while $\{1\cdot\} = 0 = \{\mathrm{i}(h/2\pi)\cdot\}$.

If, on the other hand, the Lie algebras (50*a*), (50*b*) are isomorphic and have no basis containing a nonzero constant operator, $\chi_{\dot{\jmath}} = k$, then (50*d*) defines an extension of Dirac's correspondence principle to a Lie algebraic correspondence between commutation relations of Poisson-bracket operators and commutation relations of quantum mechanical operators. When this occurs, the Lie algebras (50*a*), (50*b*), (50*d*) determine locally isomorphic quantal and classical Lie dynamical groups—groups which can be viewed as arising from geometric symmetries in classical phase space.

This is the case for the Lie algebras of the ISO(3,1) invariance groups of the relativistic systems considered herein, and the SO(4,1) and SO(4,2) Lie algebras of classical Kepler systems. For the latter, the classical dynamical group is a symplectic realization of SO(4,2) in a phase space. The SO(4,2) dynamical symmetry of the corresponding atomic system is that of the regularized Schrödinger equation which is the analog of $H = K$. Because all three Lie algebras, (50*a*), (50*b*), (50*d*), are in this case isomorphic, the geometric symmetries of Kepler systems in phase space find physical expression in the behavior of the electron in a hydrogen atom and in the motion of massive heavenly bodies. Both display consequences of SO(4,2) geometric symmetry in phase space—despite the uncertainty principle.

8. Concluding remarks

In the previous paragraphs, a canonical transformation in PQET space has been used to relate free-particle motions, defined by a Poincaré-invariant Hamiltonian function, to accelerated motions defined by an ISO(3, 1) invariant Hamiltonian that is a Lorentz invariant analog of a regularized classical Kepler Hamiltonian. A canonical diffeomorphism in ET space has been used in defining a large c limit that relates the PQET space of the open relativistic trajectories of these systems to subspaces of classical PQE space which contain Keplerian trajectories of the same, or differing topology. Taking this limit converts the second Hamiltonian function into the regularized Hamiltonian that determines the evolution of classical Kepler systems. Motions in which the Runge–Lenz vector $\mathbf{a}(E)$ is constant were described by motions of $\mathbf{a}(-E)$, and as constrained motions of 4-vectors $\boldsymbol{\rho}$, $\boldsymbol{\pi}$ in the phase spaces PQ, PQE, PQET.

Keplerian motions of the $\boldsymbol{\rho}$, and $\boldsymbol{\pi}$ are generated by first-order derivative operators that are Lie generators of rotations, hyperbolic rotations and translations in PQE phase space. All of these motions are confined to manifolds which are defined by three invariant functions of the positions and momenta, functions with self-evident symmetries in the $\boldsymbol{\rho}$, $\boldsymbol{\pi}$ coordinate systems. The local geometrical symmetries of these Kepler manifolds in phase space are the same as the dynamical symmetries of Kepler motions—and the same as the well-known local dynamical symmetries of quantum mechanical Keplerian systems.

This correspondent raises the question, 'Under what conditions does the uncertainty principle allow geometric symmetries in the phase space of a dynamical system to be locally identical to its quantum mechanical dynamical symmetries?' The question was answered in section 7 with the aid of a 'Lie algebraic extension of Dirac's correspondence principle'. The formulation of the principle begins with the establishment of a 1:1 correspondence between a large class of quantum mechanical operators and their Poison-bracket operator counterparts. It then establishes the mathematical criterion that determines whether the corresponding quantum mechanical operators and Poisson-bracket operators are generators of Lie algebras that act isomorphically. When the criterion is satisfied, the uncertainty principle allows the physics of the quantum mechanical system, and the physics of the corresponding classical or relativistic system, to express consequences of the locally identical geometrical symmetries in phase space. As it is satisfied for the relativistic, classical and quantum mechanical systems considered herein, the dynamics of all may be considered to exhibit consequences of geometrical symmetries in phase space.

The formulation of Dirac's correspondence principle given at the beginning of section 7 has been designed to eliminate the ambiguities—'impediments to quantization'—that can develop because the usual formulation of the principle does not prescribe any ordering of non-commuting operators [39]. By also taking into account non-polynomial functions and their corresponding quantum mechanical functionals, the discussion produces a substantial extension of the range of validity of an earlier statement of the consequences of extending the correspondence principle to Poisson-bracket operators [18].

It should be noted that though the connections we establish involve a Bateman–Cunningham inversion, we have not established any connections between the SO(4,2) invariance of Maxwell's equations, and the SO(4,2) dynamical symmetries of non-relativistic, or relativistic, one-electron atoms. Also, we would call attention to the fact that the relativistic regularized Kepler system produced by the canonical transformation of section 3 is a model system in which the attracting center is fixed at the spacetime origin. For a physical two-particle relativistic system, it is subject to the same criticism that Eddington leveled against Dirac's relativistic treatment of hydrogen-like atoms [40]. It is most interesting, and significant, that in designing a non-perturbative method for circumventing this problem, Barut and Baquini directly established that relativistic hydrogen atoms possess an SO(4,2) dynamical group [41].

Though our treatment of Kepler symmetry depends upon the group theory of Sophus Lie, rather than the tensor analysis of Tulio Levi Civita, it may be considered to be an application of concepts of geometrodynamics [42]. From section 3 on, no interactive forces or potentials have been assumed; the effects of the Lorentz invariant generalization of a $1/r$ potential upon the motions of a free particle have been produced geometrically by mappings which convert a Poincaré-invariant Hamiltonian function into an ISO(3,1) invariant analog of a regularized classical Kepler Hamiltonian. We have not addressed any physical questions associated with geometrodynamic interpretations of these mappings as passive rather than active transformations. This would require a geometrodynamic treatment of phase spaces. It would also require the extension of Einstein's interpretation of the consequences of the ISO(3,1) invariance of Maxwell's equations to an analogous passive interpretation of the consequences of their inversion invariance. In this connection, readers are referred to Kastrup's reviews [15].

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